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Parameter-invariant second-order variational problems in one variable

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Abstract. A projection is defined such that a second-order Lagrangian density factors through this projection modulo contact forms if and only if it is parameter invariant. In this way, a geometric interpretation of the parameter invariance conditions is obtained. The above projection is then used to prove the strict factorization of the Poincaré–Cartan form attached to a parameter-invariant variational problem thus leading us to state the Hamilton–Cartan formalism, the complete description of symmetries and regularity for such problems. The case of the squared curvature Lagrangian in the plane is analysed especially.

1. Introduction

First-order variational problems whose action integral does not change under arbitrary transformations of the independent variable (or *parameter-invariant problems*) have a significant role in pseudo-Riemannian and Finsler geometry as well as in classical and relativistic mechanics [1, 3, 12, 14, 17, 26, 27]. As is well known, first-order parameter-invariant Lagrangian densities are defined by Lagrangian functions on the tangent bundle which are positively homogeneous of first degree, and there is a standard procedure, coming back from Jacobi and Carathéodory (e.g. see [8, section 8.1.2] and references therein, or [27]), which allows one to associate a non-parametric Lagrangian to each first-order parameter-invariant problem. Let us sketch a brief review of this theory: let $\pi_{10}\colon J^1(\mathbb{R}, M) \to \mathbb{R} \times M$, be the bundle of 1-jets of curves on an n-dimensional smooth manifold M. A Lagrangian $\mathcal{L}\colon J^1(\mathbb{R}, M) \to \mathbb{R}$ is *parameter-invariant* if its fundamental integral is invariant under arbitrary changes of the parameter $\phi\colon [a, b] \to [\alpha, \beta]$ of class C^1 with $\phi'(t) > 0$ (see formula (2.1) below when $r = 1$). If we denote by $(t, x^i, \dot{x}^i)$ the local coordinates induced in $\pi_{10}^{-1}(U)$ by a local coordinate system (U, x^i) of the manifold M; i.e. $\dot{x}^k(j^1_t\sigma) = (\mathrm{d}/\mathrm{d}t)(x^k \circ \sigma)(t)$, then we can characterize first-order parameter-invariant Lagrangians as the functions $\mathcal{L}\colon J^1(\mathbb{R}, M) \to \mathbb{R}$ verifying $(\partial\mathcal{L}/\partial t) = 0$, $\dot{x}^i(\partial\mathcal{L}/\partial\dot{x}^i) = \mathcal{L}$. Taking into account the natural identification $J^1(\mathbb{R}, M) = \mathbb{R} \times TM$, the first condition above tells us that $\mathcal{L}$ can be considered as a function on TM, which is homogeneous of the first order according to the second condition (cf [8, section 8.1.1, 17, theorems 8.2 and section 8.3, 27, section 3.1]). In [8, 30], for example, a procedure to pass from a parameter-invariant (or *parametric*) Lagrangian of the first order to an associated non-parametric

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Lagrangian with the same extremals is given. If $\mathcal{L}\colon T\mathbb{R}^{n+1} \to \mathbb{R}$ is a parameter-invariant Lagrangian, the associated non-parametric Lagrangian $\bar{\mathcal{L}}\colon J^1(\mathbb{R}, \mathbb{R}^n) \to \mathbb{R}$ is just defined by $\bar{\mathcal{L}}(j^1_{x_0}\sigma) = \mathcal{L}((1_{\mathbb{R}}, \sigma)_*(\mathrm{d}/\mathrm{d}x)_{x_0})$, for every curve $\sigma\colon \mathbb{R} \to \mathbb{R}^n$.

Second-order parameter-invariant Lagrangian densities are characterized by the so-called Zermelo conditions [17, 31]. For a generalization of the Zermelo conditions to higher-order problems we refer the reader to Kawaguchi [14]. Zermelo conditions do not have such an immediate interpretation as the homogeneity condition for first-order Lagrangians. Nevertheless, second-order problems present some interesting examples in geometry and elasticity, such as those defined by $\phi(\kappa)\mathrm{d}s$ [5, 13, 24], where κ stands for the curvature of a planar curve and s is the arc-length parameter. Specially well known is the problem defined by the squared curvature leading to elastica and spline curves [6, 16, 19].

The goal of this paper is to develop a general procedure valid for parameter-invariant variational problems of first and second order which includes the classical method in the particular case of first-order problems. The major difficulty that arises in dealing with parameter-invariant variational problems is their singularity, which does not allow us to define the Hamiltonian formalism directly. In the first-order case, a standard way of introducing the Hamiltonian function for a parameter-invariant problem is given, for example, in [8, 30], but it does not seem possible to generalize this method to higher-order Lagrangians. Moreover, the general method to construct the Hamiltonian formalism for higher-order singular variational problems, using constraints and the Dirac formalism (see [4, 25, 29]) does not fit as well in the present case as it does not take into account the specific properties of these problems; i.e. the Zermelo conditions. It seems to us that our method is more natural since it allows us to introduce the Hamiltonian formulation for second-order parameter-invariant variational problems whose Hessian is of maximal rank, by defining the Hamiltonian for the associated non-parametric Lagrangian, solving its Hamilton equations and then reparametrizing the solutions arbitrarily.

Let us briefly explain the basic motivation. Let $\mathcal{L}\colon J^2(\mathbb{R}, N) \to \mathbb{R}$ be a parameter-invariant Lagrangian. If a curve $\sigma\colon \mathbb{R} \to N$ is immersive at t_0, we can consider a coordinate system $(x, y^1, \ldots, y^n)$ on N such that the velocity of σ at the point $\sigma(t_0)$ is given by $(\partial/\partial x)_{\sigma(t_0)}$. Note that the immersive character of σ at t_0 only depends on $j^1_{t_0}\sigma$ and also that immersive r-jets are a dense open subset in $J^r(\mathbb{R}, N)$ for every $r \geqslant 1$. Hence, at least locally, we can assume that the manifold N splits into a product $N = \mathbb{R} \times M$, where $\mathbb{R}$ represents the x-axis and M stands for the $(y^1, \ldots, y^n)$-manifold. Under this representation, σ is given by a pair of functions $\sigma = (f, g)$, with $f\colon \mathbb{R} \to \mathbb{R}$, $g\colon \mathbb{R} \to M$, where in addition $f'(t_0) \neq 0$ by virtue of the immersive character of the curve. Accordingly, we can associate the M-valued 'non-parametric' curve $g \circ f^{-1}\colon \mathbb{R} \to M$ to the given curve $\sigma = (f, g)\colon \mathbb{R} \to \mathbb{R} \times M$.

In section 3 the corresponding 2-jet version of the above assignment allows us to obtain a submersion of jet bundles—called the fundamental projection—through which parameter-invariant Lagrangians factor modulo contact forms in a sense made precise in theorem 4.1. In fact, this submersion is defined on a dense open subset $\mathcal{O}_r$ of the r-jet bundle $J^r(\mathbb{R}, \mathbb{R} \times M)$ for arbitrary r, although we only apply it to variational problems in the case $r = 2$. The fundamental projection is the basic tool in order to develop the Hamiltonian formalism for parameter-invariant problems. It is an important fact to remark that the Poincaré–Cartan form of a parameter-invariant Lagrangian behaves even better than the Lagrangian itself with respect to factoring through the fundamental projection, as in section 5 we prove that the Poincaré–Cartan form of a parameter-invariant Lagrangian of second order, factors through the projection. Then, after recalling the Hamilton–Cartan formalism in section 6, we obtain the theorem 6.1, which says that the extremals of a parameter-invariant Lagrangian are the extremals of its projection, composed with any local diffeomorphism of the real line. The properties of the projection regarding symmetries are studied in section 7. There we explain why the concept of generalized symmetry (cf [24, definition 5.25]) must be introduced. The behaviour of the projection with respect to regularity is analysed in section 8, where it is proved (theorem 8.2) that the rank of the Hessian of a Lagrangian density is not affected by parameter elimination. Finally, we study an example of application of all these techniques in section 9.

2. Zermelo conditions

We define the parameter invariance for higher-order Lagrangians in a similar way to the first-order case. Let M be a C^∞ manifold and let $\pi_{r0}\colon J^r(\mathbb{R}, M) \to \mathbb{R}\times M$ be the bundle of r-jets of curves in M. A Lagrangian $\mathcal{L}\colon J^r(\mathbb{R}, M) \to \mathbb{R}$ is said to be invariant under parameter transformations (or parameter invariant) if for every diffeomorphism $\phi\colon [a, b] \to [\alpha, \beta]$ of class C^r with positive derivative everywhere and each curve $\sigma\colon [\alpha, \beta] \to M$ we have

$$\int_a^b \mathcal{L}(j_u^r(\sigma \circ \phi))\,\mathrm{d}u = \int_\alpha^\beta \mathcal{L}(j_t^r(\sigma))\,\mathrm{d}t. \tag{2.1}$$

Let us now introduce some notation for the case of second-order Lagrangians: $(t, x^k; \dot{x}^k, \ddot{x}^k)$, $1 \leqslant k \leqslant n = \dim M$, stand for the coordinates induced on $\pi_{20}^{-1}(\mathbb{R} \times U)$ from a coordinate open domain $(U; x^k)$ on the manifold M; i.e.

$$\dot{x}^k(j_t^2\sigma) = \frac{\mathrm{d}}{\mathrm{d}t}(x^k \circ \sigma)(t), \ddot{x}^k(j_t^2\sigma) = \frac{\mathrm{d}^2}{\mathrm{d}t^2}(x^k \circ \sigma)(t).$$

Theorem 2.1. A second-order Lagrangian $\mathcal{L}\colon J^2(\mathbb{R}, M) \to \mathbb{R}$ is parameter invariant if and only if it satisfies the *Zermelo conditions*; i.e.

$$\frac{\partial \mathcal{L}}{\partial t} = 0 \tag{2.2}$$

$$\dot{x}^k \frac{\partial \mathcal{L}}{\partial \dot{x}^k} + 2\ddot{x}^k \frac{\partial \mathcal{L}}{\partial \ddot{x}^k} = \mathcal{L} \tag{2.3}$$

$$\dot{x}^k \frac{\partial \mathcal{L}}{\partial \ddot{x}^k} = 0. \tag{2.4}$$

For the proof, we refer the reader to [14, 17, theorem 8.5, 31].

3. The fundamental projection

In the following sections we consider curves in $\mathbb{R} \times M$, where M is an arbitrary n-dimensional manifold. We denote by $(x; y^i)$ the coordinates induced on $\mathbb{R} \times U$ from the natural coordinate x in the real line and a coordinate open domain $(U; y^i)$ in M. Let us denote by $\mathcal{O}_r \subset J^r(\mathbb{R}, \mathbb{R} \times M)$ the dense open subset defined by

$$\mathcal{O}_r = \{j_t^r\sigma \in J^r(\mathbb{R}, \mathbb{R} \times M) : \dot{x}(j_t^r\sigma) \neq 0\}.$$

Note that $\mathcal{O}_r = \pi_{r,1}^{-1}(\mathcal{O}_1)$. A curve $\sigma\colon \mathbb{R} \to \mathbb{R} \times M$ is determined by two maps $f\colon \mathbb{R} \to \mathbb{R}$, $g\colon \mathbb{R} \to M$, so that $\sigma = (f, g)$, and we can define a projection

$$p_r\colon \mathcal{O}_r \to J^r(\mathbb{R}, M),\ p_r(j_t^r\sigma) = j_{f(t)}^r(g \circ f^{-1}). \tag{3.1}$$

Lemma 3.1. The mapping p_r is a surjective submersion.

Proof. The surjectivity of p_r is trivial, as $\forall j^r_x g \in J^r(\mathbb{R}, M)$ is $j^r_x g = p_r(j^r_x(1_{\mathbb{R}}, g))$ and obviously $j^r_x(1_{\mathbb{R}}, g) \in \mathcal{O}_r$. To see that p_r is a submersion, we must check the surjectivity of

$$(p_r)_*: T_{j^r_x(f,g)}\mathcal{O}_r = T_{j^r_x(f,g)} J^r(\mathbb{R}, \mathbb{R} \times M) \to T_{j^r_{f(x)}(g\circ f^{-1})} J^r(\mathbb{R}, M).$$

We shall prove this by induction on r. For $r = 0$, it is immediate. For $r = 1$, p_1 is given by $(t, x, y^i; \dot{x}, \dot{y}^i) \mapsto (x, y^i; (y^i)' = \dot{y}^i/\dot{x})$, which is clearly a submersion, where $(x, y^i; (y^i)')$ are the coordinates induced on $\pi^{-1}_{10}(\mathbb{R} \times U)$ from a coordinate open domain $(U; y^k)$ on M. Let $r \geqslant 2$. Assume $(p_{r-1})_*$ is surjective. If we consider the map $\pi_{r,r-1}$: $J^r(\mathbb{R}, \mathbb{R} \times M) \to J^{r-1}(\mathbb{R}, \mathbb{R} \times M)$ given by $\pi_{r,r-1}(j^r_x\sigma) = j^{r-1}_x\sigma$, and the similarly defined map $\tilde{\pi}_{r,r-1}$: $J^r(\mathbb{R}, M) \to J^{r-1}(\mathbb{R}, M)$, then the following diagram is commutative:

$$\begin{array}{ccc} \mathcal{O}_r & \xrightarrow{p_r} & J^r(\mathbb{R}, M) \\ \pi_{r,r-1}\downarrow & & \downarrow \tilde{\pi}_{r,r-1} \\ \mathcal{O}_{r-1} & \xrightarrow{p_{r-1}} & J^{r-1}(\mathbb{R}, M). \end{array}$$

Hence we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \to & \ker(\pi_{r,r-1})_* & \xrightarrow{i} & TJ^r(\mathbb{R}, \mathbb{R} \times M) & \xrightarrow{(\pi_{r,r-1})_*} & TJ^{r-1}(\mathbb{R}, \mathbb{R} \times M) & \to & 0 \\ & & \downarrow (p_r)_* & & \downarrow (p_r)_* & & \downarrow (p_{r-1})_* & & \\ 0 & \to & \ker(\tilde{\pi}_{r,r-1})_* & \xrightarrow{i} & TJ^r(\mathbb{R}, M) & \xrightarrow{(\tilde{\pi}_{r,r-1})_*} & TJ^{r-1}(\mathbb{R}, M) & \to & 0 \end{array}$$

and from the snake diagram [2, proposition 2.10], we obtain the following exact sequence:

$$\begin{aligned} &0 \to \ker((p_r)_*|_{\ker(\pi_{r,r-1})_*}) \to \ker(p_r)_* \to \ker(p_{r-1})_* \\ &\to \operatorname{coker}((p_r)_*|_{\ker(\pi_{r,r-1})_*}) \to \operatorname{coker}(p_r)_* \to \operatorname{coker}(p_{r-1})_* \to 0. \end{aligned}$$

By virtue of the hypothesis we have $\operatorname{coker}(p_{r-1})_* = 0$ and we thus conclude that if $\operatorname{coker}((p_r)_*|_{\ker(\pi_{r,r-1})_*}) = 0$, then $\operatorname{coker}(p_r)_*$ will also vanish. Hence, in order to finish our proof it is enough to check the surjectivity of the mapping

$$(p_r)_*|_{(\ker(\pi_{r,r-1})_*)_{j^r_t(f,g)}}: (\ker(\pi_{r,r-1})_*)_{j^r_t(f,g)} \to (\ker(\tilde{\pi}_{r,r-1})_*)_{j^r_{f(t)}(g\circ f^{-1})}.$$

Moreover, $\ker(\pi_{r,r-1})_*$ is generated by $(\partial/\partial x^{(r)}, \partial/\partial y^{k(r)})$, and $\ker(\tilde{\pi}_{r,r-1})_*$ is generated by $\partial/\partial y^{k[r]}$, where $x^{(\alpha)}(j^r_t\sigma) = (\mathrm{d}^\alpha(x \circ \sigma)/\mathrm{d}t^\alpha)(t)$, $y^{k(\alpha)}(j^r_t\sigma) = (\mathrm{d}^\alpha(y^k \circ \sigma)/\mathrm{d}t^\alpha)(t)$, $\sigma: \mathbb{R} \to \mathbb{R} \times M$, $1 \leqslant \alpha \leqslant r$, $1 \leqslant k \leqslant n$, stand for the coordinates induced on $\pi^{-1}_{r0}(\mathbb{R} \times \mathbb{R} \times U)$ from the coordinate open domain $(\mathbb{R} \times U; x, y^k)$ on $\mathbb{R} \times M$, and $y^{k[\alpha]}(j^r_x g) = (\mathrm{d}^\alpha(y^k \circ g)/\mathrm{d}x^\alpha)(x)$, $g: \mathbb{R} \to M$, $1 \leqslant \alpha \leqslant r$, are the coordinates induced on $\tilde{\pi}^{-1}_{r0}(\mathbb{R} \times U)$ from the coordinate open domain $(U; y^k)$ on M. Hence we only need to calculate the dependence of the local expression of p_r on the highest-order derivatives. We claim that

$$y^{k[r]} \circ p_r(t, x, y^i; x^{(\alpha)}, y^{i(\alpha)}) = \frac{x^{(1)}y^{k(r)} - x^{(r)}y^{k(1)}}{(x^{(1)})^{r+1}} + F_k(x, y^i; x^{(\beta)}, y^{i(\beta)})$$
$$1 \leqslant i \leqslant n \qquad 1 \leqslant \alpha \leqslant r \qquad 1 \leqslant \beta \leqslant r-1 \qquad \forall r \geqslant 2 \tag{3.2}$$

F_k being certain functions on $J^{r-1}(\mathbb{R}, \mathbb{R} \times M)$. To prove this fact, by induction on r we state

$$\begin{aligned} \frac{\mathrm{d}^r}{\mathrm{d}x^r}(y^k \circ (g \circ f^{-1})) &= \left(\frac{(\mathrm{d}^r g^k/\mathrm{d}t^r)(\mathrm{d}f/\mathrm{d}t) - (\mathrm{d}g^k/\mathrm{d}t)(\mathrm{d}^r f/\mathrm{d}t^r)}{(\mathrm{d}f/\mathrm{d}t)^{r+1}} \circ f^{-1}\right) \\ &\quad + (\text{terms involving derivatives of order} < r) \qquad \forall r \geqslant 2 \end{aligned} \tag{3.3}$$

(where $g^k = y^k \circ g$). For $r = 2$, we have

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}(y^k \circ g \circ f^{-1})(x)\frac{\mathrm{d}}{\mathrm{d}x}\left[\left(\frac{\mathrm{d}}{\mathrm{d}t}(y^k \circ g)\right)(f^{-1}(x))\frac{\mathrm{d}}{\mathrm{d}x}(f^{-1})(x)\right](x)$$

$$= \frac{\mathrm{d}}{\mathrm{d}x}\left[\left(\frac{\mathrm{d}g^k}{\mathrm{d}t}\frac{1}{(\mathrm{d}f/\mathrm{d}t)}\right)\circ f^{-1}\right](x)$$

$$= \left[\frac{(\mathrm{d}^2 g^k/\mathrm{d}t^2)(\mathrm{d}f/\mathrm{d}t) - (\mathrm{d}g^k/\mathrm{d}t)(\mathrm{d}^2 f/\mathrm{d}t^2)}{(\mathrm{d}f/\mathrm{d}t)^2}\frac{1}{(\mathrm{d}f/\mathrm{d}t)}\right](f^{-1}(x)).$$

Now, if we suppose that (3.3) holds true for $r-1$, then we have

$$\frac{\mathrm{d}^r}{\mathrm{d}x^r}(y^k \circ (g \circ f^{-1}))(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{(\mathrm{d}^{r-1} g^k/\mathrm{d}t^{r-1})(\mathrm{d}f/\mathrm{d}t) - (\mathrm{d}g^k/\mathrm{d}t)(\mathrm{d}^{r-1} f/\mathrm{d}t^{r-1})}{(\mathrm{d}f/\mathrm{d}t)^r}\circ f^{-1}\right.$$

$$\left. + \text{(terms involving derivatives of order} < r-1)\right](x)$$

$$= \left[\frac{(\mathrm{d}^r g^k/\mathrm{d}t^r)(\mathrm{d}f/\mathrm{d}t) - (\mathrm{d}g^k/\mathrm{d}t)(\mathrm{d}^r f/\mathrm{d}t^r) + \text{(terms of order} < r)}{(\mathrm{d}f/\mathrm{d}t)^r}\right.$$

$$\left.\times\frac{1}{(\mathrm{d}f/\mathrm{d}t)}\circ f^{-1} + \text{(terms involving derivatives of order} < r)\right](x)$$

$$= \left(\frac{(\mathrm{d}^r g^k/\mathrm{d}t^r)(\mathrm{d}f/\mathrm{d}t) - (\mathrm{d}g^k/\mathrm{d}t)(\mathrm{d}^r f/\mathrm{d}t^r)}{(\mathrm{d}f/\mathrm{d}t)^{r+1}}\circ f^{-1}\right)(x)$$

$$+ \text{(terms involving derivatives of order} < r)$$

thus proving formula (3.3). Using this formula, we obtain

$$y^{k[r]} \circ p_r(j^r_t(f,g))y^{k[r]}(j^r_{f(t)}(g \circ f^{-1})) = \frac{\mathrm{d}^r}{\mathrm{d}x^r}(y^k \circ (g \circ f^{-1}))(f(t))$$

$$= \left(\frac{(\mathrm{d}^r g^k/\mathrm{d}t^r)(\mathrm{d}f/\mathrm{d}t) - (\mathrm{d}g^k/\mathrm{d}t)(\mathrm{d}^r f/\mathrm{d}t^r)}{(\mathrm{d}f/\mathrm{d}t)^{r+1}}\right)(t)$$

$$+ \text{(terms involving derivatives of order} < r)$$

$$= \left(\frac{x^{(1)}y^{k(r)} - x^{(r)}y^{k(1)}}{(x^{(1)})^{r+1}} + F_k(x, y^i; x^{(\beta)}, y^{i(\beta)})\right)(j^r_t(f,g))$$

$$1 \leqslant i \leqslant n \qquad 1 \leqslant \alpha \leqslant r \qquad 1 \leqslant \beta \leqslant r-1 \qquad \forall r \geqslant 2$$

thus proving our claim. Finally, from the formula (3.2) we derive the local expression for $(p_r)_*|_{(\ker(\pi_{r,r-1})_*)_{j^r_t(f,g)}}$; i.e.

$$(p_r)_*\left(\frac{\partial}{\partial x^{(r)}}\right) = -\frac{y^{k(1)}}{(x^{(1)})^{r+1}}\frac{\partial}{\partial y^{k[r]}} \qquad (p_r)_*\left(\frac{\partial}{\partial y^{k(r)}}\right) = \frac{1}{(x^{(1)})^r}\frac{\partial}{\partial y^{k[r]}}$$

which shows that it is a surjective map. □

Remark 3.1. It is a well known fact (e.g. see [10, section 5]) that the canonical projection $\pi_{r,r-1}\colon J^r(\mathbb{R}, M) \to J^{r-1}(\mathbb{R}, M)$ admits an affine bundle structure modelled over the vector bundle $((pr_1 \circ \pi_{10})^* S^k T^*\mathbb{R}) \otimes ((pr_2 \circ \pi_{10})^* TM)$, where $pr_1\colon \mathbb{R} \times M \to \mathbb{R}$, $pr_2\colon \mathbb{R} \times M \to M$ are the projections onto the factors. Taking this construction into account, it is proved that the map $p_r\colon \mathcal{O}_r \subset J^r(\mathbb{R}, \mathbb{R} \times M) \to J^r(\mathbb{R}, M)$ is an affine bundle morphism over p_{r-1}.

4. Factoring invariant Lagrangians

In this section, we obtain the connection between the projection p_2 and the second-order parameter-invariant Lagrangians. From now on we consider Lagrangians which are defined on the dense open subset $\mathcal{O}_2 \subset J^2(\mathbb{R}, \mathbb{R} \times M)$. Recall that $\mathcal{O}_r = \pi_{r,1}^{-1}(\mathcal{O}_1)$ (cf section 3), so that $\mathcal{O}_r$ is the set of all r-jets whose velocity has a non-vanishing x-component. Also note that $\mathcal{O}_2$ is natural under changes of parameter; i.e. if $j_t^2\sigma \in \mathcal{O}_2$, then $j^2_{\phi^{-1}(t)}(\sigma \circ \phi)$ also belongs to $\mathcal{O}_2$ for every diffeomorphism ϕ.

Theorem 4.1. A second-order Lagrangian $\mathcal{L}: \mathcal{O}_2 \to \mathbb{R}$ is invariant under parameter transformations if and only if the Lagrangian density $\mathcal{L}\mathrm{d}t$ factors through p_2 modulo contact forms; i.e. there exists $\bar{\mathcal{L}}: J^2(\mathbb{R}, M) \to \mathbb{R}$ such that $p_2^*(\bar{\mathcal{L}}\mathrm{d}x) = \mathcal{L}\mathrm{d}t + \eta$, where η is a contact form in $J^2(\mathbb{R}, \mathbb{R} \times M)$. $\bar{\mathcal{L}}$ is called the *non-parametric Lagrangian* associated with $\mathcal{L}$.

Proof. Let t and x be two global coordinate systems on $\mathbb{R}$, and (y^i) a local coordinate system on M. We consider on $\mathcal{O}_2 \subset J^2(\mathbb{R}, \mathbb{R} \times M)$ the coordinates $(t, x, y^i; \dot{x}, \dot{y}^i, \ddot{x}, \ddot{y}^i)$, given by t, x and y^i; and on $J^2(\mathbb{R}, M)$ the coordinates $(x, y^i;\ (y^i)', (y^i)'')$ given by x, y^i. As a first step in the proof, let us see that $\mathcal{L}$ is parameter-invariant if and only if $(1/\dot{x})\mathcal{L}$ factors through p_2, i.e. if there is an $\bar{\mathcal{L}}: J^2(\mathbb{R}, M) \to \mathbb{R}$ such that

$$(1/\dot{x})\mathcal{L} = p_2^*(\bar{\mathcal{L}}) = \bar{\mathcal{L}} \circ p_2. \tag{4.1}$$

The local expression of p_2 is $p_2(t, x, y^i; \dot{x}, \dot{y}^i, \ddot{x}, \ddot{y}^i) = (x, y^i; (y^i)' = \dot{y}^i/\dot{x}, (y^i)'' = (\dot{x}\ddot{y}^i - \dot{y}^i\ddot{x})/\dot{x}^3)$. So p_{2*} has the following local expression:

$$\begin{aligned}
&p_{2*}\left(\frac{\partial}{\partial t}\right) = 0 \qquad p_{2*}\left(\frac{\partial}{\partial x}\right) = \frac{\partial}{\partial x} \qquad p_{2*}\left(\frac{\partial}{\partial y^i}\right) = \frac{\partial}{\partial y^i}\\
&p_{2*}\left(\frac{\partial}{\partial \dot{x}}\right) = -\frac{\dot{y}^j}{\dot{x}^2}\frac{\partial}{\partial (y^j)'} + \left(-\frac{2\ddot{y}^j}{\dot{x}^3} + \frac{3\dot{y}^j\ddot{x}}{\dot{x}^4}\right)\frac{\partial}{\partial (y^j)''}\\
&p_{2*}\left(\frac{\partial}{\partial \dot{y}^i}\right) = \frac{1}{\dot{x}}\frac{\partial}{\partial (y^i)'} - \frac{\ddot{x}}{\dot{x}^3}\frac{\partial}{\partial (y^i)''}\\
&p_{2*}\left(\frac{\partial}{\partial \ddot{x}}\right) = -\frac{\dot{y}^j}{\dot{x}^3}\frac{\partial}{\partial (y^j)''} \qquad p_{2*}\left(\frac{\partial}{\partial \ddot{y}^i}\right) = \frac{1}{\dot{x}^2}\frac{\partial}{\partial (y^i)''}.
\end{aligned} \tag{4.2}$$

Hence, ker p_{2*} is generated by

$$\frac{\partial}{\partial t}, \phi = \dot{x}\frac{\partial}{\partial \dot{x}} + \dot{y}^i\frac{\partial}{\partial \dot{y}^i} + 2\ddot{x}\frac{\partial}{\partial \ddot{x}} + 2\ddot{y}^i\frac{\partial}{\partial \ddot{y}^i} \qquad \chi = \dot{x}\frac{\partial}{\partial \ddot{x}} + \dot{y}^i\frac{\partial}{\partial \ddot{y}^i}.$$

As p_2 is a surjective submersion with connected fibres, the necessary and sufficient condition for a function to factor through p_2 is that the Lie derivative of the function in the direction of any vector field in ker p_{2*} vanishes. Hence

$$0 = \mathrm{L}_{\frac{\partial}{\partial t}}\left(\frac{1}{\dot{x}}\mathcal{L}\right) = \frac{1}{\dot{x}}\frac{\partial \mathcal{L}}{\partial t} \iff \frac{\partial \mathcal{L}}{\partial t} = 0 \tag{4.3}$$

$$\begin{aligned}
0 = \mathrm{L}_{\phi}\left(\frac{1}{\dot{x}}\mathcal{L}\right) &= -\frac{1}{\dot{x}}\mathcal{L} + \frac{1}{\dot{x}}\left(\dot{x}\frac{\partial \mathcal{L}}{\partial \dot{x}} + \dot{y}^i\frac{\partial \mathcal{L}}{\partial \dot{y}^i} + 2\ddot{x}\frac{\partial \mathcal{L}}{\partial \ddot{x}} + 2\ddot{y}^i\frac{\partial \mathcal{L}}{\partial \ddot{y}^i}\right)\\
&\iff \dot{x}\frac{\partial \mathcal{L}}{\partial \dot{x}} + \dot{y}^i\frac{\partial \mathcal{L}}{\partial \dot{y}^i} + 2\ddot{x}\frac{\partial \mathcal{L}}{\partial \ddot{x}} + 2\ddot{y}^i\frac{\partial \mathcal{L}}{\partial \ddot{y}^i} = \mathcal{L}
\end{aligned} \tag{4.4}$$

$$0 = \mathrm{L}_{\chi}\left(\frac{1}{\dot{x}}\mathcal{L}\right) = \frac{1}{\dot{x}}\left(\dot{x}\frac{\partial \mathcal{L}}{\partial \ddot{x}} + \dot{y}^i\frac{\partial \mathcal{L}}{\partial \ddot{y}^i}\right) \iff \dot{x}\frac{\partial \mathcal{L}}{\partial \ddot{x}} + \dot{y}^i\frac{\partial \mathcal{L}}{\partial \ddot{y}^i} = 0. \tag{4.5}$$

As (4.3)–(4.5) coincide with the Zermelo conditions (2.2)–(2.4), respectively, we conclude that parameter invariance is equivalent to (4.1).

Now, let us suppose that (4.1) is fulfilled. As (4.2) yields $p_2^*(\mathrm{d}x) = \mathrm{d}x$, we deduce $p_2^*(\bar{\mathcal{L}}\mathrm{d}x) = p_2^*(\bar{\mathcal{L}})\mathrm{d}x = \frac{1}{\dot{x}}\mathcal{L}\mathrm{d}x = \frac{1}{\dot{x}}\mathcal{L}(\mathrm{d}x - \dot{x}\mathrm{d}t) + \mathcal{L}\mathrm{d}t = \mathcal{L}\mathrm{d}t + p_2^*(\bar{\mathcal{L}})(\mathrm{d}x - \dot{x}\mathrm{d}t)$; i.e. $\mathcal{L}\mathrm{d}t$ factors through p_2 modulo contact forms. Conversely, let us suppose that there exists a $\bar{\mathcal{L}}\colon J^2(\mathbb{R}, M) \to \mathbb{R}$ such that $p_2^*(\bar{\mathcal{L}}\mathrm{d}x) = \mathcal{L}\mathrm{d}t + \eta$, η being a contact form. Hence $p_2^*(\bar{\mathcal{L}})\mathrm{d}x = p_2^*(\bar{\mathcal{L}}\mathrm{d}x) = \mathcal{L}\mathrm{d}t + a_0(\mathrm{d}x - \dot{x}\mathrm{d}t) + a_1(\mathrm{d}\dot{x} - \ddot{x}\mathrm{d}t) + \sum b_{0,i}(\mathrm{d}y^i - \dot{y}^i\mathrm{d}t) + \sum b_{1,i}(\mathrm{d}\dot{y}^i - \ddot{y}^i\mathrm{d}t)$, and taking components we obtain $a_1 = b_{0,i} = b_{1,i} = 0$, $\forall i$, $a_0 = p_2^*\bar{\mathcal{L}}$, and so $\mathcal{L} - \dot{x}\,p_2^*(\bar{\mathcal{L}}) = 0$; i.e. (4.1) holds and $\mathcal{L}$ is parameter invariant. □

5. Factoring Poincaré–Cartan forms

In section 4 we have seen that the map p_2 allows us to 'eliminate' the parameter from a parameter-invariant Lagrangian, modulo contact forms. For Poincaré–Cartan forms the result is even better as the Poincaré–Cartan form of $\mathcal{L}\mathrm{d}t$ is exactly p_3-projectable onto the Poincaré–Cartan form of $\bar{\mathcal{L}}\mathrm{d}x$.

As is well known (e.g. see [28, theorem 2.1]) an n-form $\Theta(\mathcal{L}\mathrm{d}t)$ on $J^{2r-1}(\mathbb{R}, M)$ (the *Poincaré–Cartan form*) is associated to each rth-order Lagrangian density $\mathcal{L}\mathrm{d}t$ on $J^r(\mathbb{R}, M)$, whose local expression is

$$\Theta(\mathcal{L}\mathrm{d}t) = \mathcal{L}\mathrm{d}t + \sum_{h=1}^{n}\sum_{\alpha=0}^{r-1}\left(\sum_{i=0}^{r-1-\alpha}(-1)^i(D_t)^i\left(\frac{\partial\mathcal{L}}{\partial x^{h(\alpha+i+1)}}\right)\right)\theta_\alpha^h \qquad (5.1)$$

where $\theta_\alpha^h = \mathrm{d}y^{h(\alpha)} - y^{h(\alpha+1)}\mathrm{d}t$ are the standard contact forms on $J^r(\mathbb{R}, M)$ (e.g. see [28]), and D_t is the total derivation operator; i.e. the $\mathbb{R}$-derivation $D_t\colon C^\infty(J^k(\mathbb{R}, M)) \to C^\infty(J^{k+1}(\mathbb{R}, M))$, $\forall k \in \mathbb{N}$, given by $(D_t(f))(j_t^{k+1}\sigma) = (\mathrm{d}(f \circ j^k\sigma)/\mathrm{d}t)(t)$, $f \in C^\infty(J^r(\mathbb{R}, M))$, whose local expression is

$$D_t = \frac{\partial}{\partial t} + \sum_{h=1}^{n}\sum_{\alpha=0}^{\infty} x^{h(\alpha+1)}\frac{\partial}{\partial x^{h(\alpha)}}.$$

Theorem 5.1. Let $\mathcal{L}\colon \mathcal{O}_2 \to \mathbb{R}$ be a parameter-invariant Lagrangian, and let $\bar{\mathcal{L}}$ be as in theorem 4.1. The Poincaré–Cartan form of $\mathcal{L}\mathrm{d}t$ factors through p_3 onto the Poincaré–Cartan form of $\bar{\mathcal{L}}\mathrm{d}x$, i.e. $p_3^*(\Theta(\bar{\mathcal{L}}\mathrm{d}x)) = \Theta(\mathcal{L}\mathrm{d}t)$.

Proof. The local expression of p_3 agrees with that of p_2 up to order 2, and $(y^i)''' = (\dot{x}^2\dddot{y}^i - 3\dot{x}\ddot{x}\ddot{y}^i + 3\dot{y}^i\ddot{x}^2 - \dot{x}\dot{y}^i\dddot{x})/\dot{x}^5$. Taking into account that in our case $\mathcal{L}$ is defined on $\mathcal{O}_2 \subset J^2(\mathbb{R}, \mathbb{R} \times M)$, recalling the way in which coordinates are induced on this bundle and the condition (2.2), we obtain,

$$\begin{aligned}\Theta(\mathcal{L}\mathrm{d}t) = \mathcal{L}\mathrm{d}t &+ \frac{\partial\mathcal{L}}{\partial\dot{x}}(\mathrm{d}x - \dot{x}\mathrm{d}t) + \frac{\partial\mathcal{L}}{\partial\dot{y}^h}(\mathrm{d}y^h - \dot{y}^h\mathrm{d}t) - D_t\left(\frac{\partial\mathcal{L}}{\partial\ddot{x}}\right)(\mathrm{d}x - \dot{x}\mathrm{d}t)\\ &- D_t\left(\frac{\partial\mathcal{L}}{\partial\ddot{y}^h}\right)(\mathrm{d}y^h - \dot{y}^h\mathrm{d}t) + \frac{\partial\mathcal{L}}{\partial\ddot{x}}(\mathrm{d}\dot{x} - \ddot{x}\mathrm{d}t) + \frac{\partial\mathcal{L}}{\partial\ddot{y}^h}(\mathrm{d}\dot{y}^h - \ddot{y}^h\mathrm{d}t).\end{aligned} \qquad (5.2)$$

Application of the total derivative D_t in the formula (2.4) yields

$$\ddot{x}\frac{\partial\mathcal{L}}{\partial\ddot{x}} + \ddot{y}^i\frac{\partial\mathcal{L}}{\partial\ddot{y}^i} + \dot{x}D_t\left(\frac{\partial\mathcal{L}}{\partial\ddot{x}}\right) + \dot{y}^iD_t\left(\frac{\partial\mathcal{L}}{\partial\ddot{y}^i}\right) = 0.$$

Back substitution of the above expression into (5.2) gives as a result that

$$\begin{aligned}\Theta(\mathcal{L}\mathrm{d}t) &= \mathcal{L}\mathrm{d}t + \frac{\partial\mathcal{L}}{\partial\dot{x}}(\mathrm{d}x - \dot{x}\mathrm{d}t) + \frac{\partial\mathcal{L}}{\partial\dot{y}^h}(\mathrm{d}y^h - \dot{y}^h\mathrm{d}t) - D_t\left(\frac{\partial\mathcal{L}}{\partial\ddot{x}}\right)\mathrm{d}x \\ &\quad - D_t\left(\frac{\partial\mathcal{L}}{\partial\ddot{y}^h}\right)\mathrm{d}y^h - \ddot{x}\frac{\partial\mathcal{L}}{\partial\ddot{x}}\mathrm{d}t - \ddot{y}^i\frac{\partial\mathcal{L}}{\partial\ddot{y}^i}\mathrm{d}t + \frac{\partial\mathcal{L}}{\partial\ddot{x}}(\mathrm{d}\dot{x} - \ddot{x}\mathrm{d}t) + \frac{\partial\mathcal{L}}{\partial\ddot{y}^h}(\mathrm{d}\dot{y}^h - \ddot{y}^h\mathrm{d}t) \\ &\overset{(2.3)}{=} \frac{\partial\mathcal{L}}{\partial\dot{x}}\mathrm{d}x + \frac{\partial\mathcal{L}}{\partial\dot{y}^h}\mathrm{d}y^h + \frac{\partial\mathcal{L}}{\partial\ddot{x}}\mathrm{d}\dot{x} + \frac{\partial\mathcal{L}}{\partial\ddot{y}^h}\mathrm{d}\dot{y}^h - D_t\left(\frac{\partial\mathcal{L}}{\partial\ddot{x}}\right)\mathrm{d}x - D_t\left(\frac{\partial\mathcal{L}}{\partial\ddot{y}^h}\right)\mathrm{d}y^h.\end{aligned} \tag{5.3}$$

The local version of formula (4.1) states (recall that $\mathcal{L}$ does not depend on t, by virtue of (2.2))

$$\mathcal{L}(x, y^i; \dot{x}, \dot{y}^i, \ddot{x}, \ddot{y}^i) = \dot{x}\bar{\mathcal{L}}\left(x, y^i; \frac{\dot{y}^i}{\dot{x}}, \frac{\dot{x}\ddot{y}^i - \dot{y}^i\ddot{x}}{\dot{x}^3}\right) \tag{5.4}$$

and so (with the abuse of notation of writing $\bar{\mathcal{L}}$ instead of $p_3^*\bar{\mathcal{L}} = p_2^*\bar{\mathcal{L}} = \bar{\mathcal{L}} \circ p_2$)

$$\begin{aligned}\frac{\partial\mathcal{L}}{\partial\dot{x}} &= \bar{\mathcal{L}} - \frac{\dot{y}^h}{\dot{x}}\frac{\partial\bar{\mathcal{L}}}{\partial(y^h)'} + \frac{-2\dot{x}\ddot{y}^h + 3\dot{y}^h\ddot{x}}{\dot{x}^3}\frac{\partial\bar{\mathcal{L}}}{\partial(y^h)''} \\ \frac{\partial\mathcal{L}}{\partial\dot{y}^h} &= \frac{\partial\bar{\mathcal{L}}}{\partial(y^h)'} - \frac{\ddot{x}}{\dot{x}^2}\frac{\partial\bar{\mathcal{L}}}{\partial(y^h)''} \\ \frac{\partial\mathcal{L}}{\partial\ddot{x}} &= -\frac{\dot{y}^h}{\dot{x}^2}\frac{\partial\bar{\mathcal{L}}}{\partial(y^h)''} \qquad \frac{\partial\mathcal{L}}{\partial\ddot{y}^h} = \frac{1}{\dot{x}}\frac{\partial\bar{\mathcal{L}}}{\partial(y^h)''}.\end{aligned} \tag{5.5}$$

Back substitution of these values in (5.3) yields

$$\begin{aligned}\Theta(\mathcal{L}\mathrm{d}t) &= \bar{\mathcal{L}}\mathrm{d}x - \frac{\dot{y}^h}{\dot{x}}\frac{\partial\bar{\mathcal{L}}}{\partial(y^h)'}\mathrm{d}x + \frac{-2\dot{x}\ddot{y}^h + 3\dot{y}^h\ddot{x}}{\dot{x}^3}\frac{\partial\bar{\mathcal{L}}}{\partial(y^h)''}\mathrm{d}x + \frac{\partial\bar{\mathcal{L}}}{\partial(y^h)'}\mathrm{d}y^h - \frac{\ddot{x}}{\dot{x}^2}\frac{\partial\bar{\mathcal{L}}}{\partial(y^h)''}\mathrm{d}y^h \\ &\quad - \frac{\dot{y}^h}{\dot{x}^2}\frac{\partial\bar{\mathcal{L}}}{\partial(y^h)''}\mathrm{d}\dot{x} + \frac{1}{\dot{x}}\frac{\partial\bar{\mathcal{L}}}{\partial(y^h)''}\mathrm{d}\dot{y}^h - D_t\left(\frac{\partial\mathcal{L}}{\partial\ddot{x}}\right)\mathrm{d}x - D_t\left(\frac{\partial\mathcal{L}}{\partial\ddot{y}^h}\right)\mathrm{d}y^h.\end{aligned} \tag{5.6}$$

We expand the terms with total derivatives, using (2.2), (5.5) and its derivatives whenever it is necessary, thus obtaining

$$D_t\left(\frac{\partial\mathcal{L}}{\partial\ddot{x}}\right) = -(y^h)'D_x\left(\frac{\partial\bar{\mathcal{L}}}{\partial(y^h)''}\right) + \frac{2\dot{y}^h\ddot{x} - \dot{x}\ddot{y}^h}{\dot{x}^3}\frac{\partial\bar{\mathcal{L}}}{\partial(y^h)''}. \tag{5.7}$$

Similar calculations lead us to

$$D_t\left(\frac{\partial\mathcal{L}}{\partial\ddot{y}^h}\right) = D_x\left(\frac{\partial\bar{\mathcal{L}}}{\partial(y^h)''}\right) - \frac{\ddot{x}}{\dot{x}^2}\frac{\partial\bar{\mathcal{L}}}{\partial(y^h)''}. \tag{5.8}$$

By back substitution of the expressions (5.7) and (5.8) in (5.6), we obtain

$$\begin{aligned}\Theta(\mathcal{L}\mathrm{d}t) &= \bar{\mathcal{L}}\mathrm{d}x - \frac{\dot{y}^h}{\dot{x}}\frac{\partial\bar{\mathcal{L}}}{\partial(y^h)'}\mathrm{d}x + \frac{-\dot{x}\ddot{y}^h + \dot{y}^h\ddot{x}}{\dot{x}^3}\frac{\partial\bar{\mathcal{L}}}{\partial(y^h)''}\mathrm{d}x + \frac{\partial\bar{\mathcal{L}}}{\partial(y^h)'}\mathrm{d}y^h \\ &\quad + \left(-\frac{\dot{y}^h}{\dot{x}^2}\mathrm{d}\dot{x} + \frac{1}{\dot{x}}\mathrm{d}\dot{y}^h\right)\frac{\partial\bar{\mathcal{L}}}{\partial(y^h)''} + (y^h)'D_x\left(\frac{\partial\bar{\mathcal{L}}}{\partial(y^h)''}\right)\mathrm{d}x - D_x\left(\frac{\partial\bar{\mathcal{L}}}{\partial(y^h)''}\right)\mathrm{d}y^h\end{aligned}$$

which yields (taking into account that $p_3^*(\mathrm{d}(y^i)') = -(\dot{y^i}/\dot{x}^2)\mathrm{d}\dot{x} + (1/\dot{x})\mathrm{d}\dot{y^i}$, and that $\bar{\mathcal{L}}$ is written in the place of $p_2^*\bar{\mathcal{L}} = p_3^*\bar{\mathcal{L}}$),

$$\Theta(\mathcal{L}\mathrm{d}t) = p_3^*\bar{\mathcal{L}}p_3^*\mathrm{d}x + p_3^*\frac{\partial\bar{\mathcal{L}}}{\partial(y^h)'}(p_3^*\mathrm{d}y^h - (y^h)'p_3^*\mathrm{d}x) + p_3^*\frac{\partial\bar{\mathcal{L}}}{\partial(y^h)''}(p_3^*\mathrm{d}(y^h)' - (y^h)''p_3^*\mathrm{d}x)$$
$$-p_3^*D_x\left(\frac{\partial\bar{\mathcal{L}}}{\partial(y^h)''}\right)(p_3^*\mathrm{d}y^h - (y^h)'p_3^*\mathrm{d}x) \overset{(5.1)}{=} p_3^*\Theta(\bar{\mathcal{L}}\mathrm{d}x).$$

□

6. The Hamilton–Cartan formulation

As is well known , a curve $\sigma: \mathbb{R} \to M$ is an extremal of the variational problem defined by an rth order Lagrangian $\mathcal{L}\mathrm{d}t$ if and only if for every vector field $X \in \mathfrak{X}(J^{2r-1}(\mathbb{R}, M))$ we have

$$(j^{2r-1}\sigma)^*(i_X\mathrm{d}\Theta(\mathcal{L}\mathrm{d}t)) = 0. \tag{6.1}$$

It suffices that (6.1) holds true for pr_1-vertical vector fields (cf [11, equation (3.7), 28]). The above equation is called the *Hamilton–Cartan equation* and it is on the basis of the Hamiltonian formalism. In fact, we have

$$(j^{2r-1}\sigma)^*(i_X\mathrm{d}\Theta(\mathcal{L}\mathrm{d}t)) = \sum_{j=0}^{r}(-1)^j\frac{\mathrm{d}^j}{\mathrm{d}t^j}\left(\frac{\partial\mathcal{L}}{\partial x^{h(j)}}\circ j^r\sigma\right)\theta_0^h(X)(j^{2r-1}\sigma)\mathrm{d}t.$$

Equation (6.1) and theorem 5.1 lead us to the following characterization of extremals of a second-order parameter-invariant Lagrangian.

Theorem 6.1. Let $\mathcal{L}$ be a second-order parameter-invariant Lagrangian on $\tilde{pr}_1 : \mathbb{R}\times(\mathbb{R}\times M) \to \mathbb{R}$. If $(f, g): \mathbb{R} \to \mathbb{R}\times M$ is an extremal of $\mathcal{L}\mathrm{d}t$ such that $j^2(f, g) \in \mathcal{O}_2$, then $h = g\circ f^{-1}: \mathbb{R} \to M$ is an extremal of $\bar{\mathcal{L}}\mathrm{d}x$. Conversely, if $h: \mathbb{R} \to M$ is an extremal of $\bar{\mathcal{L}}\mathrm{d}x$, then for every local diffeomorphism $f: \mathbb{R} \to \mathbb{R}$ the curve $(f, h\circ f)$ is an extremal of $\mathcal{L}\mathrm{d}t$.

Proof. Let $h: \mathbb{R} \to M$ be a curve on M. As we showed in lemma 3.1, p_3 is surjective, the fibre over j^3h being $\{j^3(f, h\circ f) : \forall f$ local diffeomorphism$\}$. Using this fact and theorem 5.1 for every $X \in \mathfrak{X}(J^3(\mathbb{R}, \mathbb{R}\times M))$ we obtain

$$(j^3h)^*i_{(p_3)_*X}\mathrm{d}\Theta(\bar{\mathcal{L}}\mathrm{d}x) = (j^3(f, h\circ f))^*p_3^*i_{(p_3)_*X}\mathrm{d}\Theta(\bar{\mathcal{L}}\mathrm{d}x)$$
$$= (j^3(f, h\circ f))^*i_X\mathrm{d}\Theta(\mathcal{L}\mathrm{d}t).$$

Taking into account the surjectivity of p_{3*} (again by lemma 3.1), the above formula proves that h is an extremal of $\bar{\mathcal{L}}\mathrm{d}x$ if and only if $(f, h\circ f)$ is an extremal of $\mathcal{L}\mathrm{d}t$. As every (f, g) such that $j^2(f, g) \in \mathcal{O}_2$ can be written as $(f, (g\circ f^{-1})\circ f)$, the proof is complete. □

Remark 6.1. The above theorem means that the extremals of a second-order parameter-invariant Lagrangian density $\mathcal{L}\mathrm{d}t$ whose velocity has a nowhere vanishing x-component can be obtained from the extremals of the non-parametric Lagrangian $\bar{\mathcal{L}}\mathrm{d}x$, by composing them with an arbitrary local diffeomorphism of the real line.

7. Infinitesimal symmetries

Let $\mathcal{L}: J^r(\mathbb{R}, M) \to \mathbb{R}$ be an rth order Lagrangian. A pr_1-projectable vector field $X \in \mathfrak{X}(\mathbb{R} \times M)$ is said to be an *infinitesimal symmetry* of $\mathcal{L}\mathrm{d}t$ if $\mathrm{L}_{X_{(r)}}(\mathcal{L}\mathrm{d}t) = 0$. In such a case, Noether's theorem [17, 18, 22, 23] states that the function $f_X = i_{X_{(2r-1)}}\Theta(\mathcal{L}\mathrm{d}t): J^{2r-1}(\mathbb{R}, M) \to \mathbb{R}$ is constant on the extremals of the variational problem associated with the Lagrangian $\mathcal{L}$, where $X_{(2r-1)}$ is the prolongation of the vector field X to $J^{2r-1}(\mathbb{R}, M)$ by means of infinitesimal contact transformations (cf [15, 20]). The function f_X is called the *Noether invariant* associated with X. Let us consider a more general situation and define a *generalized infinitesimal symmetry* of $\mathcal{L}\mathrm{d}t$ as a vector field $X \in \mathfrak{X}(J^0(\mathbb{R}, M)) = \mathfrak{X}(\mathbb{R} \times M)$ (not necessarily pr_1-projectable) such that $\mathrm{L}_{X_{(r)}}(\mathcal{L}\mathrm{d}t)$ is a contact form; i.e. it vanishes on every r-jet of curve on M (cf [24, definition 5.25]).

Remark 7.1. If $X \in \mathfrak{X}(\mathbb{R} \times M)$ is a generalized infinitesimal symmetry of $\mathcal{L}\mathrm{d}t$, then f_X is constant on the extremals of $\mathcal{L}\mathrm{d}t$, i.e. generalized symmetries also produce Noether invariants.

Proof. From the hypothesis we have $\mathrm{L}_{X_{(r)}}(\mathcal{L}\mathrm{d}t) = \theta$, where θ is a contact form. As $\Theta(\mathcal{L}\mathrm{d}t) = \mathcal{L}\mathrm{d}t+$ contact forms, and $X_{(2r-1)}$ is an infinitesimal contact transformation, we obtain $\mathrm{L}_{X_{(2r-1)}}\Theta(\mathcal{L}\mathrm{d}t) = \eta$, η being a contact form. Thus, $\eta = i_{X_{(2r-1)}}\mathrm{d}\Theta(\mathcal{L}\mathrm{d}t) + \mathrm{d}i_{X_{(2r-1)}}\Theta(\mathcal{L}\mathrm{d}t) = i_{X_{(2r-1)}}\mathrm{d}\Theta(\mathcal{L}\mathrm{d}t) + \mathrm{d}f_X$. If $\sigma: \mathbb{R} \to M$ is an extremal of $\mathcal{L}\mathrm{d}t$, then $(j^{2r-1}\sigma)^* i_{X_{(2r-1)}}\mathrm{d}\Theta(\mathcal{L}\mathrm{d}t) + \mathrm{d}(f_X \circ j^{2r-1}\sigma) = (j^{2r-1}\sigma)^*\eta = 0$, as η is a contact form; but the first summand of the above expression vanishes by virtue of the Hamilton equation (6.1), so we conclude that $f_X \circ j^{2r-1}\sigma$ is constant; i.e. f_X is constant on extremals. □

To prove the results in this section, we shall make use of the following lemma.

Lemma 7.1. For $X \in \mathfrak{X}(\mathbb{R} \times M)$, let $X_{(r)}$ and $X_{[r]}$ be the prolongations of X by infinitesimal contact transformations to $J^r(\mathbb{R}, \mathbb{R} \times M)$ and to $J^r(\mathbb{R}, M)$, respectively. Then $X_{[r]} = p_{r*}(X_{(r)})$.

Proof. Let $\tilde{pr}_1$ and $\tilde{pr}_2$ be the canonical projections of $\mathbb{R} \times (\mathbb{R} \times M)$ onto $\mathbb{R}$ and $\mathbb{R} \times M$, respectively. If X is the infinitesimal generator of a one-parameter group of transformations (Φ_s, ϕ_s) of the fibration $pr_1: \mathbb{R} \times M \to \mathbb{R}$, $X_{[r]}$ will be the infinitesimal generator of the one-parameter group $\Phi_s^{[r]}$ of $J^r(\mathbb{R}, M)$, given by

$$\Phi_s^{[r]}(j_t^r h) = j_{\phi_s(t)}^r(pr_2 \circ \Phi_s \circ (1_{\mathbb{R}}, h) \circ \phi_{-s}).$$

Moreover, X can be identified to the vector field $\tilde{X} = (0, X) \in \mathfrak{X}(\mathbb{R} \times (\mathbb{R} \times M))$ (as a matter of fact, $\tilde{X}$ is $\tilde{pr}_1$-vertical and $\tilde{pr}_2$-projectable, and X is its projection onto $\mathbb{R} \times M$ by $\tilde{pr}_2$). $\tilde{X}$ is the infinitesimal generator of the one-parameter group $(\tilde{\Phi}_s, 1_{\mathbb{R}})$ of the fibration $\tilde{pr}_1: \mathbb{R} \times (\mathbb{R} \times M) \to \mathbb{R}$, where $\tilde{\Phi}_s = (1_{\mathbb{R}}, \Phi_s)$, and $X_{(r)}$ is the generator of the one-parameter group $\Phi_s^{(r)}$ of $J^r(\mathbb{R} \times (\mathbb{R} \times M))$, given by $\Phi_s^{(r)}(j_t^r(f, g)) = j_t^r(\tilde{pr}_2 \circ \tilde{\Phi}_s \circ (1_{\mathbb{R}}, (f, g)) \circ 1_{\mathbb{R}}) = j_t^r(\Phi_s(f, g)) = j_t^r(\phi_s \circ f, pr_2 \circ \Phi_s \circ (f, g))$. Hence, $p_r \circ \Phi_s^{(r)}(j_t^r(f, g)) = \Phi_s^{[r]}\left(j_{f(t)}^r(g \circ f^{-1})\right) = \Phi_s^{[r]} \circ p_r\left(j_t^r(f, g)\right)$, $\forall j_t^r(f, g) \in U$, and we conclude $p_r \circ \Phi_s^{(r)} = \Phi_s^{[r]} \circ p_r$; i.e. $p_{r*}X_{(r)} = X_{[r]}$. □

Lemma 7.1 allows us to prove that if $\mathcal{L}$ is a second-order parameter-invariant Lagrangian on $\mathbb{R} \times M$, a vector field on $\mathbb{R} \times M$ is an infinitesimal symmetry of $\Theta(\mathcal{L}\mathrm{d}t)$ (viewed as the projection of a $\tilde{pr}_1$-vertical, $\tilde{pr}_2$-projectable field on $\mathbb{R} \times (\mathbb{R} \times M)$) if and only if it is a generalized infinitesimal symmetry of $\Theta(\bar{\mathcal{L}}\mathrm{d}x)$ with vanishing associated contact form. More precisely,

Theorem 7.2. Let $\mathcal{L}\colon \mathcal{O}_2 \to \mathbb{R}$ be a parameter-invariant Lagrangian, and let $X \in \mathfrak{X}(\mathbb{R}\times M)$. Then, $\mathrm{L}_{X_{(3)}}\Theta(\mathcal{L}\mathrm{d}t) = 0$ if and only if $\mathrm{L}_{X_{[3]}}\Theta(\bar{\mathcal{L}}\mathrm{d}x) = 0$.

Proof. Theorem 5.1 and lemma 7.1 yield

$$\mathrm{L}_{X_{(3)}}\Theta(\mathcal{L}\mathrm{d}t) = \mathrm{L}_{X_{(3)}}p_3^*\Theta(\bar{\mathcal{L}}\mathrm{d}x) = p_3^*\mathrm{L}_{p_{3*}X_{(3)}}\Theta(\bar{\mathcal{L}}\mathrm{d}x) = p_3^*\mathrm{L}_{X_{[3]}}\Theta(\bar{\mathcal{L}}\mathrm{d}x) \tag{7.1}$$

and the result follows as p_3^* is injective (or, equivalently, p_{3*}, is surjective, recalling lemma 3.1). □

The next theorem provides us with a relationship between some infinitesimal symmetries of a parameter-invariant Lagrangian $\mathcal{L}$ and the generalized infinitesimal symmetries of its non-parametric Lagrangian $\bar{\mathcal{L}}$. Again, this result is weaker than the one that we have just obtained for symmetries of the Poincaré–Cartan form.

Theorem 7.3. Let $\mathcal{L}\colon \mathcal{O}_2 \to \mathbb{R}$ be a parameter-invariant Lagrangian and $\bar{\mathcal{L}}$ as in theorem 4.1. We identify each $\tilde{pr}_2$-projectable vector field $X \in \mathfrak{X}(\mathbb{R}\times M)$ to the $\tilde{pr}_1$-vertical vector field $(0, X) \in \mathfrak{X}(\mathbb{R}\times\mathbb{R}\times M)$. Then, X is an infinitesimal symmetry of $\mathcal{L}\mathrm{d}t$ if and only if X is a generalized infinitesimal symmetry of $\bar{\mathcal{L}}\mathrm{d}x$.

Proof. Let (x, y^i) be local coordinates in $\mathbb{R}\times M$, and let us write the field X as $X = \alpha(x, y^i)(\partial/\partial x) + \beta^i(x, y^i)(\partial/\partial y^i)$. Then (see [20]),

$$X_{(2)} = \alpha\frac{\partial}{\partial x} + \beta^i\frac{\partial}{\partial y^i} + \alpha_1\frac{\partial}{\partial \dot{x}} + \beta_1^i\frac{\partial}{\partial \dot{y}^i} + \alpha_2\frac{\partial}{\partial \ddot{x}} + \beta_2^i\frac{\partial}{\partial \ddot{y}^i} \tag{7.2}$$

where

$$\alpha_1 = \frac{\partial\alpha}{\partial x}\dot{x} + \frac{\partial\alpha}{\partial y^j}\dot{y}^j \tag{7.3}$$

(the other coefficients will not be used in the proof). We begin with the direct implication. From the hypothesis, $\mathrm{L}_{X_{(2)}}(\mathcal{L}\mathrm{d}t) = (X_{(2)}(\mathcal{L}))\mathrm{d}t = 0$. We thus obtain

$$\begin{aligned}
p_2^*\mathrm{L}_{X_{[2]}}(\bar{\mathcal{L}}\mathrm{d}x) &= p_2^*\mathrm{L}_{p_{2*}X_{(2)}}(\bar{\mathcal{L}}\mathrm{d}x) = p_2^* i_{p_{2*}X_{(2)}}\mathrm{d}(\bar{\mathcal{L}}\mathrm{d}x) + p_2^*\mathrm{d}i_{p_{2*}X_{(2)}}(\bar{\mathcal{L}}\mathrm{d}x)\\
&= i_{X_{(2)}}\mathrm{d}(p_2^*(\bar{\mathcal{L}}\mathrm{d}x)) + \mathrm{d}i_{X_{(2)}}p_2^*(\bar{\mathcal{L}}\mathrm{d}x)\\
&\overset{(4.1)}{=} i_{X_{(2)}}\mathrm{d}\left(\frac{1}{\dot{x}}\mathcal{L}\mathrm{d}x\right) + \mathrm{d}i_{X_{(2)}}\left(\frac{1}{\dot{x}}\mathcal{L}\mathrm{d}x\right)\\
&= i_{X_{(2)}}\left(-\frac{1}{\dot{x}^2}\mathcal{L}\mathrm{d}\dot{x}\wedge\mathrm{d}x + \frac{1}{\dot{x}}\mathrm{d}\mathcal{L}\wedge\mathrm{d}x\right) + \mathrm{d}\left(\frac{1}{\dot{x}}\mathcal{L}i_{X_{(2)}}(\mathrm{d}x)\right)\\
&\overset{(7.2)}{=} -\frac{1}{\dot{x}^2}\mathcal{L}\alpha_1\mathrm{d}x + \frac{1}{\dot{x}^2}\mathcal{L}\alpha\mathrm{d}\dot{x} + \frac{1}{\dot{x}}i_{X_{(2)}}(\mathrm{d}\mathcal{L})\mathrm{d}x - \frac{1}{\dot{x}}\alpha\mathrm{d}\mathcal{L}\\
&\quad -\frac{1}{\dot{x}^2}\mathcal{L}\alpha\mathrm{d}\dot{x} + \frac{1}{\dot{x}}\alpha\mathrm{d}\mathcal{L} + \frac{1}{\dot{x}}\mathcal{L}\mathrm{d}\alpha\\
&= \frac{1}{\dot{x}}\left(X_{(2)}(\mathcal{L})\mathrm{d}x + \mathcal{L}\mathrm{d}\alpha - \frac{1}{\dot{x}}\mathcal{L}\alpha_1\mathrm{d}x\right)\\
&\overset{(7.3)}{=} \frac{1}{\dot{x}}\mathcal{L}\left[\frac{\partial\alpha}{\partial x}\mathrm{d}x + \frac{\partial\alpha}{\partial y^i}\mathrm{d}y^i - \frac{1}{\dot{x}}\left(\frac{\partial\alpha}{\partial x}\dot{x} + \frac{\partial\alpha}{\partial y^i}\dot{y}^i\right)\mathrm{d}x\right]\\
&= \frac{1}{\dot{x}}\mathcal{L}\frac{\partial\alpha}{\partial y^i}\left(\mathrm{d}y^i - \frac{\dot{y}^i}{\dot{x}}\mathrm{d}x\right) = p_2^*\left(\bar{\mathcal{L}}\frac{\partial\alpha}{\partial y^i}(\mathrm{d}y^i - (y^i)'\mathrm{d}x)\right).
\end{aligned}$$

As p_2^* is injective (or, equivalently, p_{2*} is surjective; see lemma 3.1) we conclude $\mathrm{L}_{X_{[2]}}(\bar{\mathcal{L}}\mathrm{d}x) = \bar{\mathcal{L}}(\partial\alpha/\partial y^i)(\mathrm{d}y^i - (y^i)'\mathrm{d}x)$; i.e. it is a contact form, and so X is a generalized infinitesimal symmetry of $\bar{\mathcal{L}}\mathrm{d}x$.

Conversely, let us suppose that $\mathrm{L}_{X_{[2]}}(\bar{\mathcal{L}}\mathrm{d}x)$ is a contact form. As $\Theta(\phi)$ is congruent with ϕ modulo contact forms for every density ϕ (see (5.1)), we have that $\Theta(\mathrm{L}_{X_{[2]}}(\bar{\mathcal{L}}\mathrm{d}x))$ is a contact form. We recall that the infinitesimal functoriality of Poincaré–Cartan form (e.g. see [21, theorem 2]) means $\mathrm{L}_{X_{(3)}}\Theta(\mathcal{L}\mathrm{d}t) = \Theta(\mathrm{L}_{X_{(2)}}(\mathcal{L}\mathrm{d}t))$. Using this fact and the formula (7.1) we obtain

$$\begin{aligned} p_3^*\Theta(\mathrm{L}_{X_{[2]}}(\bar{\mathcal{L}}\mathrm{d}x)) &= p_3^*\mathrm{L}_{X_{[3]}}\Theta(\bar{\mathcal{L}}\mathrm{d}x) = \mathrm{L}_{X_{(3)}}\Theta(\mathcal{L}\mathrm{d}t) \\ &= \Theta(\mathrm{L}_{X_{(2)}}(\mathcal{L}\mathrm{d}t)) \overset{(2.2)}{=} \Theta(X_{(2)}(\mathcal{L})\mathrm{d}t) \\ &\overset{(5.1)}{=} X_{(2)}(\mathcal{L})\mathrm{d}t + \text{contact forms} \end{aligned}$$

and from the local expression of p_3,

$$\begin{aligned} p_3^*(\mathrm{d}y^i - (y^i)'\mathrm{d}x) &= \mathrm{d}y^i - \frac{\dot{y}^i}{\dot{x}}\mathrm{d}x = (\mathrm{d}y^i - \dot{y}^i\mathrm{d}t) - \frac{\dot{y}^i}{\dot{x}}(\mathrm{d}x - \dot{x}\mathrm{d}t) \\ p_3^*(\mathrm{d}(y^i)' - (y^i)''\mathrm{d}x) &= -\frac{\dot{y}^i}{\dot{x}^2}(\mathrm{d}\dot{x} - \ddot{x}\mathrm{d}t) + \frac{1}{\dot{x}}(\mathrm{d}\dot{y}^i - \ddot{y}^i\mathrm{d}t) - \frac{\dot{x}\ddot{y}^i - \dot{y}^i\ddot{x}}{\dot{x}^3}(\mathrm{d}x - \dot{x}\mathrm{d}t) \\ p_3^*(\mathrm{d}(y^i)'' - (y^i)'''\mathrm{d}x) &= \frac{-2\dot{x}\ddot{y}^i + 3\dot{y}^i\ddot{x}}{\dot{x}^4}(\mathrm{d}\dot{x} - \ddot{x}\mathrm{d}t) + \frac{\ddot{x}}{\dot{x}^3}(\mathrm{d}\dot{y}^i - \ddot{y}^i\mathrm{d}t) - \frac{\dot{y}^i}{\dot{x}^3}(\mathrm{d}\ddot{x} - \dddot{x}\mathrm{d}t) \\ &\quad + \frac{1}{\dot{x}^2}(\mathrm{d}\ddot{y}^i - \dddot{y}^i\mathrm{d}t) - \frac{\dot{x}^2\dddot{y}^i - 3\dot{x}\ddot{x}\ddot{y}^i + 3\dot{y}^i\ddot{x}^2 - \dot{x}\dot{y}^i\dddot{x}}{\dot{x}^5}(\mathrm{d}x - \dot{x}\mathrm{d}t) \end{aligned}$$

i.e. p_3^* maps contact forms onto contact forms, and so we have that $X_{(2)}(\mathcal{L})\mathrm{d}t$ is a contact form. Hence $X_{(2)}(\mathcal{L}) = 0$; i.e. $\mathrm{L}_{X_{(2)}}(\mathcal{L}\mathrm{d}t) = 0$. □

This theorem shows us how p_2 projects some infinitesimal symmetries of a second-order parameter-invariant Lagrangian onto the generalized infinitesimal symmetries of its non-parametric Lagrangian. Then, the next consequence of lemma 7.1 and theorem 5.1 is that p_3 maps the Noether invariants onto the corresponding Noether invariants.

Corollary 7.4. Let $\mathcal{L}\colon \mathcal{O}_2 \to J^2(\mathbb{R}, M)$ be a parameter-invariant Lagrangian, let $\bar{\mathcal{L}}$ be the corresponding non-parametric Lagrangian and let X be a vector field on $\mathfrak{X}(\mathbb{R} \times M)$. Let $f_{\tilde{X}}\colon J^3(\mathbb{R}, \mathbb{R} \times M) \to \mathbb{R}$, $f_X\colon J^3(\mathbb{R}, M) \to \mathbb{R}$ be the functions $f_{\tilde{X}} = i_{X_{(3)}}\Theta(\mathcal{L}\mathrm{d}t)$, $f_X = i_{X_{[3]}}\Theta(\bar{\mathcal{L}}\mathrm{d}x)$, respectively. Then, $f_{\tilde{X}}|_{\mathcal{O}_3} = f_X \circ p_3$.

Remark 7.2. The vector field $\partial/\partial t$ is an infinitesimal symmetry for every second-order parameter-invariant Lagrangian, by (2.2). We could thus expect to obtain some information from its Noether invariant, the *Hamiltonian*. In the case that we 'eliminated' the parameter, this information would be lost, as $p_{r*}(\partial/\partial t) = 0$. In fact, no useful information can be obtained from the Hamiltonian of a parameter-invariant variational problem, as it vanishes identically.

Proof. If $\mathcal{L}(t, x^k; \dot{x}^k, \ddot{x}^k)$ is a second-order parameter-invariant Lagrangian, the Hamiltonian is $H = -i_{\partial/\partial t}\Theta(\mathcal{L}\mathrm{d}t)$ and (2.2), (5.1) yield

$$H = -\mathcal{L} + \frac{\partial\mathcal{L}}{\partial\dot{x}^k}\dot{x}^k - D_t\left(\frac{\partial\mathcal{L}}{\partial\ddot{x}^k}\right)\dot{x}^k + \frac{\partial\mathcal{L}}{\partial\ddot{x}^k}\ddot{x}^k. \tag{7.4}$$

Applying the total derivation in (2.4), we obtain $D_t(\partial\mathcal{L}/\partial\ddot{x}^k)\dot{x}^k + \partial\mathcal{L}/\ddot{x}^k = 0$, and finally, by back substitution in (7.4), we have

$$H = -\mathcal{L} + \frac{\partial\mathcal{L}}{\partial\dot{x}^k}\dot{x}^k + 2\frac{\partial\mathcal{L}}{\partial\ddot{x}^k}\ddot{x}^k \overset{(2.3)}{=} 0.$$

□

8. The Hessian metric: regularity

Let $q\colon N \to M$ be an affine bundle modelled over the vector bundle $p\colon V \to M$ and let $f\colon N \to \mathbb{R}$ be a differentiable function. There is a canonical isomorphism $q^*V \cong \ker q_*$ given by the map $(y, v) \mapsto X_{y,v}$, where

$$X_{y,v}(f) = \lim_{t\to 0}\frac{f(tv+y) - f(y)}{t} \qquad f \in C^\infty(N)$$

is the directional derivative of f at $y \in N$, in the direction of $v \in V_{q(y)}$ (see [9]). For $x_0 \in M$, let $f_{x_0}\colon V_{x_0} \times N_{x_0} \to \mathbb{R}$ be the function given by $f_{x_0}(v, y) = X_{y,v}(f)$, $\forall v \in V_{x_0}$, $\forall y \in N_{x_0}$. For each $y_0 \in N_{x_0}$, we define $\mathrm{Hess}_{y_0}(f)\colon V_{x_0} \times V_{x_0} \to \mathbb{R}$ as

$$\mathrm{Hess}_{y_0}(f)(v, w) = X_{y_0,w}(f_{x_0}(v, \cdot)) \qquad \forall v, w \in V_{x_0}$$

where $f_{x_0}(v, \cdot)\colon N_{x_0} \to \mathbb{R}$ is the function $f_{x_0}(v, \cdot)(y) = f_{x_0}(v, y)$. Note that $\mathrm{Hess}_{y_0}(f)$ is well defined, as $X_{y_0,v}$ is a tangent vector to the fibre N_{x_0}. For more details, see [11]. Let us calculate the local expression of $\mathrm{Hess}_{y_0}(f)$. Let $(U; x^1, \dots, x^n)$ be a coordinate domain in M such that V and N trivialize on U, $(\phi^1, \dots, \phi^r)$ a basis of sections of $\Gamma(U, V)$, and $s\colon U \to N$ a section of q such that $s(x_0) = y_0$. Then, (x^j, s, ϕ^i) induces a coordinate system $(x^1, \dots, x^n; y^1, \dots, y^r)$ on $q^{-1}(U)$ as follows: $(\sum_{i=1}^r y^i(y)\phi^i(x)) + s(x) = y$, $\forall y \in q^{-1}(U)$. In these coordinates, we obtain $f_{x_0}(v, y) = \sum_{i=1}^r(\partial f/\partial y^i)(y)v^i$, $v = \sum_{i=1}^r v^i\phi^i(x_0)$, and $\mathrm{Hess}_{y_0}(f)(v, w) = \sum_{i,j=1}^r(\partial^2 f/\partial y^i\partial y^j)(y_0)v_i w_j$. Therefore $\mathrm{Hess}_{y_0}(f)$ is a symmetric bilinear form; so it defines a tensor $\mathrm{Hess}_{y_0}(f) \in S^2V^*_{x_0}$. Using the affine structure of $J^r(\mathbb{R}, M)$ (remark 3.1), we can construct the tensor $\mathrm{Hess}(\mathcal{L})$ for a Lagrangian $\mathcal{L}\colon J^2(\mathbb{R}, M) \to \mathbb{R}$, thus obtaining a metric $\mathrm{Hess}_{j^2_{t_0}\sigma}(\mathcal{L}) \in S^2(S^2(T\mathbb{R})_{t_0} \otimes_{J^1_{t_0}(\mathbb{R},M)} (T^*M)_{\sigma(t_0)})$, which is known as the *Hessian metric* of $\mathcal{L}$, and whose local expression is

$$\mathrm{Hess}(\mathcal{L}) = \sum_{i,j=1}^n \frac{\partial^2\mathcal{L}}{\partial\ddot{x}^i\partial\ddot{x}^j}\frac{\partial}{\partial t} \otimes \frac{\partial}{\partial t} \otimes \mathrm{d}x^i \otimes \frac{\partial}{\partial t} \otimes \frac{\partial}{\partial t} \otimes \mathrm{d}x^j. \tag{8.1}$$

Now consider $p_2\colon \mathcal{O}_2 \to J^2(\mathbb{R}, M)$, which is a morphism of affine bundles over p_1, (see remark 3.1) and therefore induces a vector bundle morphism

$$\boldsymbol{p}_2\colon S^2T^*\mathbb{R} \otimes_{J^1(\mathbb{R},\mathbb{R}\times M)} T(\mathbb{R} \times M) \to S^2T^*\mathbb{R} \otimes_{J^1(\mathbb{R},M)} TM$$

as follows: if h is the only element of $S^2T^*\mathbb{R}\otimes T(\mathbb{R}\times M)$ such that $j^2_t(f, g) = h + j^2_t(\bar{f}, \bar{g})$, with $j^1_t(f, g) = j^1_t(\bar{f}, \bar{g})$, then $\boldsymbol{p}_2(h)$ is the only element in $S^2T^*\mathbb{R} \otimes TM$ such that $p_2(j^2_t(f, g)) = \boldsymbol{p}_2(h) + p_2(j^2_t(\bar{f}, \bar{g}))$. In local coordinates, if $(t, x, y^i, \dot{x}, \dot{y}^i; h^0, h^i)$ represents the element

$$h = h^0(\mathrm{d}t \otimes \mathrm{d}t \otimes (\partial/\partial x))_{j^1_t(f,g)} + h^i(\mathrm{d}t \otimes \mathrm{d}t \otimes (\partial/\partial y^i))_{j^1_t(f,g)} \qquad 1 \leqslant i \leqslant n$$

$j^1_t(f, g)$ being given by $(t, x, y^i, \dot{x}, \dot{y}^i)$, then

$$\boldsymbol{p}_2(h) = \frac{h^i\dot{x}(j^1_t(f, g)) - h^0\dot{y}^i(j^1_t(f, g))}{(\dot{x}(j^1_t(f, g)))^3}(\mathrm{d}t \otimes \mathrm{d}t \otimes (\partial/\partial y^i))_{j^1_{f(t)}(g\circ f^{-1})}.$$

Accordingly, the coordinates of $\boldsymbol{p}_2(h)$ are $(x, y^i, (y^i)' = \dot{y}/\dot{x}; (h^i\dot{x} - h^0\dot{y}^i)/\dot{x}^3)$.

Theorem 8.1. Let $\mathcal{L}\colon \mathcal{O}_2 \to \mathbb{R}$ be a parameter-invariant Lagrangian, and let $\bar{\mathcal{L}}$ be the non-parametric Lagrangian associated with $\mathcal{L}$. Then,

$$\mathrm{Hess}_{j^2_{t_0}(f,g)}(\mathcal{L})(h,\bar{h}) = \frac{\mathrm{d}f}{\mathrm{d}t}(t_0)\mathrm{Hess}_{j^2_{f(t_0)}(g\circ f^{-1})}(\bar{\mathcal{L}})(\boldsymbol{p}_2(h),\boldsymbol{p}_2(\bar{h}))$$

for all $h,\bar{h} \in (S^2T^*\mathbb{R}\otimes T(\mathbb{R}\times M))_{j^1_{t_0}(f,g)}$.

Proof. From the local expression of $\boldsymbol{p}_2$ we obtain

$$\begin{aligned}\mathrm{Hess}(\bar{\mathcal{L}})(\boldsymbol{p}_2(h^0,h^i),\boldsymbol{p}_2(\bar{h^0},\bar{h^i})) &= \sum_{i,j=1}^{n}\frac{\partial^2\bar{\mathcal{L}}}{\partial(y^i)''\partial(y^j)''}\frac{1}{\dot{x}^4}h^i\bar{h^j}\\ &\quad+\sum_{j=1}^{n}\left(\sum_{i=1}^{n}\frac{\partial^2\bar{\mathcal{L}}}{\partial(y^i)''\partial(y^j)''}\frac{-\dot{y^i}}{\dot{x}^5}\right)h^0\bar{h^j}+\sum_{i=1}^{n}\left(\sum_{j=1}^{n}\frac{\partial^2\bar{\mathcal{L}}}{\partial(y^i)''\partial(y^j)''}\frac{-\dot{y^j}}{\dot{x}^5}\right)h^i\bar{h^0}\\ &\quad+\left(\sum_{i,j=1}^{n}\frac{\partial^2\bar{\mathcal{L}}}{\partial(y^i)''\partial(y^j)''}\frac{\dot{y^i}\dot{y^j}}{\dot{x}^6}\right)h^0\bar{h^0}.\end{aligned}\tag{8.2}$$

The second derivatives of $\mathcal{L}$ with respect to $(\ddot{x},\ddot{y^i})$ in formula (5.4) are

$$\begin{aligned}\frac{\partial^2\bar{\mathcal{L}}}{\partial(y^i)''\partial(y^j)''}\frac{1}{\dot{x}^3} = \frac{\partial^2\mathcal{L}}{\partial\ddot{y^i}\partial\ddot{y^j}} \qquad &\sum_i\frac{\partial^2\bar{\mathcal{L}}}{\partial(y^i)''\partial(y^j)''}\frac{-\dot{y^i}}{\dot{x}^4}\\ = \frac{\partial^2\mathcal{L}}{\partial\ddot{x}\partial\ddot{y^i}}\sum_{i,j}\frac{\partial^2\bar{\mathcal{L}}}{\partial(y^i)''\partial(y^j)''}&\frac{\dot{y^i}\dot{y^j}}{\dot{x}^5} = \frac{\partial^2\mathcal{L}}{\partial\ddot{x}^2}.\end{aligned}\tag{8.3}$$

Back substitution in (8.2) yields

$$(\mathrm{Hess}(\bar{\mathcal{L}}))(\boldsymbol{p}_2(h^0,h^i),\boldsymbol{p}_2(\bar{h^0},\bar{h^i})) = \frac{1}{\dot{x}}\mathrm{Hess}(\mathcal{L})((h^0,h^i),(\bar{h^0},\bar{h^i})).$$

□

Remark 8.1. We recall that a Lagrangian $\mathcal{L}\colon J^2(\mathbb{R},M)\to\mathbb{R}$ is said to be *regular* if the Hessian metric $\mathrm{Hess}(\mathcal{L})$ is non-degenerate. The regularity of a Lagrangian allows us to perform a change of coordinates between the variables $(t;x^k,\dot{x^k},\ddot{x^k},\dddot{x}^k)$ and the canonical variables $(t;x^k,\dot{x^k};p^k,\dot{p^k})$, where the p's are the *generalized momenta* (e.g. see [7]), also known as *Jacobi–Ostrogradski momenta*:

$$p^k = \frac{\partial\mathcal{L}}{\partial\dot{x^k}} - D_t\left(\frac{\partial\mathcal{L}}{\partial\ddot{x^k}}\right) \qquad \dot{p^k} = \frac{\partial\mathcal{L}}{\partial\ddot{x^k}}.\tag{8.4}$$

Proof. The Jacobian of the coordinate change is

$$\frac{\partial(t,x^k,\dot{x^k},p^k,\dot{p^k})}{\partial(t,x^k,\dot{x^k},\ddot{x^k},\dddot{x}^k)} = \frac{\partial(p^k,\dot{p^k})}{\partial(\ddot{x^k},\dddot{x}^k)}.$$

As

$$p^k = \frac{\partial\mathcal{L}}{\partial\dot{x^k}} - \frac{\partial^2\mathcal{L}}{\partial t\partial\ddot{x^k}} - \dot{x^i}\frac{\partial^2\mathcal{L}}{\partial x^i\partial\ddot{x^k}} - \ddot{x^i}\frac{\partial^2\mathcal{L}}{\partial\dot{x^i}\partial\ddot{x^k}} - \dddot{x}^i\frac{\partial^2\mathcal{L}}{\partial\ddot{x^i}\partial\ddot{x^k}}$$

we have that the above determinant is

$$\det\begin{pmatrix} \dfrac{\partial p^k}{\partial \ddot{x}^i} & \dfrac{\partial p^k}{\partial \dddot{x}^i} \\ \dfrac{\partial \dot{p}^k}{\partial \ddot{x}^i} & \dfrac{\partial \dot{p}^k}{\partial \dddot{x}^i} \end{pmatrix} = \det\begin{pmatrix} * & -\dfrac{\partial^2\mathcal{L}}{\partial \ddot{x}^i \ddot{x}^k} \\ \dfrac{\partial^2\mathcal{L}}{\partial \ddot{x}^i \ddot{x}^k} & 0 \end{pmatrix}$$

which is non-zero if and only if $\det(\partial^2\mathcal{L}/\partial\ddot{x}^i\ddot{x}^k) \neq 0$. □

Remark 8.2. Parameter-invariant Lagrangians of second order are not regular.

Proof. The condition on Hess($\mathcal{L}$) of being non-degenerate can be written locally as $\det(\partial^2\mathcal{L}/\partial\ddot{x}^i\partial\ddot{x}^j) \neq 0$. Derivation with respect to $\ddot{x}^i$ in (2.4) yields $\dot{x}^k(\partial^2\mathcal{L}/\partial\ddot{x}^i\partial\ddot{x}^k) = 0$, and so we obtain a vanishing linear combination of the columns (or rows) of the matrix $(\partial^2\mathcal{L}/\partial\ddot{x}^i\partial\ddot{x}^j)$ which is non-trivial at each point where a component $\dot{x}^i \neq 0$. □

The projection p_2, however, may carry a parameter-invariant (and thus non-regular) Lagrangian $\mathcal{L}$ over a regular Lagrangian $\bar{\mathcal{L}}$.

Theorem 8.2. Let $\mathcal{L}\colon\mathcal{O}_2 \to \mathbb{R}$ be a parameter-invariant Lagrangian, and let $\bar{\mathcal{L}}$ be the non-parametric Lagrangian associated with $\mathcal{L}$. Then, $\bar{\mathcal{L}}$ is regular if and only if the rank of the Hessian of $\mathcal{L}$ is $n = \dim M = \dim(\mathbb{R}\times M) - 1$.

Proof. The local expression of Hess($\mathcal{L}$) is given by the matrix

$$\begin{pmatrix} \frac{\partial^2\mathcal{L}}{\partial\ddot{x}^2} & \frac{\partial^2\mathcal{L}}{\partial\ddot{x}\ddot{y}^1} & \cdots & \frac{\partial^2\mathcal{L}}{\partial\ddot{x}\ddot{y}^n} \\ \frac{\partial^2\mathcal{L}}{\partial\ddot{x}\ddot{y}^1} & \frac{\partial^2\mathcal{L}}{\partial\ddot{y}^{1\,2}} & \cdots & \frac{\partial^2\mathcal{L}}{\partial\ddot{y}^1\ddot{y}^n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2\mathcal{L}}{\partial\ddot{x}\ddot{y}^n} & \frac{\partial^2\mathcal{L}}{\partial\ddot{y}^1\ddot{y}^n} & \cdots & \frac{\partial^2\mathcal{L}}{\partial\ddot{y}^{n\,2}} \end{pmatrix}. \tag{8.5}$$

Using the formulae (8.3), we obtain

$$\begin{aligned} \frac{\partial^2\mathcal{L}}{\partial\ddot{x}^2} &= \sum_{i,j=1}^{n}\frac{\dot{y}^i\dot{y}^j}{\dot{x}^5}\frac{\partial^2\bar{\mathcal{L}}}{\partial(y^i)''(y^j)''} = -\sum_{i=1}^{n}\dot{y}^i\dot{x}\frac{\partial^2\mathcal{L}}{\partial\ddot{x}\ddot{y}^i} \\ \frac{\partial^2\mathcal{L}}{\partial\ddot{x}\ddot{y}^j} &= \sum_{i=1}^{n}\frac{\dot{y}^i}{\dot{x}^4}\frac{\partial^2\bar{\mathcal{L}}}{\partial(y^i)''(y^j)''} = -\sum_{i=1}^{n}\dot{y}^i\dot{x}\frac{\partial^2\mathcal{L}}{\partial\ddot{y}^i\ddot{y}^j} \end{aligned} \tag{8.6}$$

i.e. the first row of the matrix (8.5) is a linear combination of the other rows, and also the first column of the above matrix is a linear combination of the other columns. So the rank of (8.5) is the same as the rank of

$$\begin{pmatrix} \frac{\partial^2\mathcal{L}}{\partial\ddot{y}^{1\,2}} & \cdots & \frac{\partial^2\mathcal{L}}{\partial\ddot{y}^1\ddot{y}^n} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2\mathcal{L}}{\partial\ddot{y}^1\ddot{y}^n} & \cdots & \frac{\partial^2\mathcal{L}}{\partial\ddot{y}^{n\,2}} \end{pmatrix}.$$

From (8.3) it follows that the above matrix represents the Hessian metric of $\mathcal{L}$ multiplied by the scalar $1/\dot{x}^3$ (recall $\dot{x} \neq 0$ as we are in $\mathcal{O}_2$). Hence, the rank of Hess($\mathcal{L}$) equals the rank of Hess($\bar{\mathcal{L}}$). □

Remark 8.3. The importance of this result lies in the fact that for a regular Lagrangian $\mathcal{L}\colon J^2(\mathbb{R}, M) \to \mathbb{R}$, the equation

$$(\gamma_3)^* i_X \mathrm{d}\Theta(\mathcal{L}\mathrm{d}t) = 0 \qquad \forall X \in \mathfrak{X}(J^3(\mathbb{R}, M)) \tag{8.7}$$

(where γ_3 is a section of $J^3(\mathbb{R}, M)$) is equivalent to the existence of an extremal of $\mathcal{L}\mathrm{d}t$, σ, such that $\gamma_3 = j^3\sigma$ (cf [11, theorem 3.1, 28]).

Hence, in order to obtain the extremals of a regular second-order Lagrangian, it suffices to solve a Pffafian system in the 3-jet bundle. If the Lagrangian is parameter invariant, it is not regular (see remark 8.2), and we cannot apply this method. Nevertheless, if the rank of the Hessian of $\mathcal{L}$ is equal to dim $(\mathbb{R} \times M) - 1$, the non-parametric Lagrangian is regular and we can obtain its extremals by solving the exterior differential system given by (8.7). Then, we transport these extremals to $\mathbb{R} \times M$ by reparametrizing them, as we saw in remark 6.1. In this way we obtain a Hamiltonian formulation for second-order parameter-invariant variational problems whose Hessian metric is of maximal rank.

9. An example

As an example of application, we calculate the Hamilton equations to the non-parametric version of the squared-curvature Lagrangian in $\mathbb{R}^2$. We consider the Lagrangian $\mathcal{L}\mathrm{d}t = \kappa_\sigma^2 \mathrm{d}s$, where κ_σ is the curvature of $\sigma(t) = (x(t), y(t))$ and s is the arc-length parameter. We have

$$\kappa_\sigma = \frac{\dot{x}\ddot{y} - \ddot{x}\dot{y}}{(\dot{x}^2 + \dot{y}^2)^{3/2}} \qquad \mathrm{d}s = (\dot{x}^2 + \dot{y}^2)^{1/2}\mathrm{d}t \qquad \mathcal{L}\mathrm{d}t = \frac{(\dot{x}\ddot{y} - \ddot{x}\dot{y})^2}{(\dot{x}^2 + \dot{y}^2)^{5/2}}\mathrm{d}t.$$

As a simple calculation shows, $\mathcal{L}$ satisfies the Zermelo conditions and hence it is parameter-invariant. After factorization through p_2, we obtain the non-parametric Lagrangian, whose expression is (cf [19, equation (3.2)]):

$$\bar{\mathcal{L}}\mathrm{d}x = \frac{y''^2}{(1 + y'^2)^{5/2}}\mathrm{d}x$$

and accordingly, it is obviously regular ($\partial^2\bar{\mathcal{L}}/\partial y''^2 \neq 0$).

Applying the formulae (8.4), we obtain the generalized momenta

$$\begin{aligned} p &= 5y'y''^2(1 + y'^2)^{-7/2} - 2y'''(1 + y'^2)^{-5/2} \\ &= (1 + y'^2)^{-7/2}(5y'y''^2 - 2y'''(1 + y'^2)) \\ p' &= 2y''(1 + y'^2)^{-5/2}. \end{aligned}$$

Using (5.1), we calculate the Poincaré–Cartan form for $\bar{\mathcal{L}}$,

$$\begin{aligned} \Theta(\bar{\mathcal{L}}\mathrm{d}x) = {} & y''^2(1 + y'^2)^{-5/2}\mathrm{d}x + 2y''(1 + y'^2)^{-5/2}(\mathrm{d}y' - y''\mathrm{d}x) \\ & + \left(5y'y''^2(1 + y'^2)^{-7/2} - 2y'''(1 + y'^2)^{-5/2}\right)(\mathrm{d}y - y'\mathrm{d}x) \end{aligned}$$

and finally, the Hamiltonian

$$\begin{aligned} H = {} & -i_{\frac{\partial}{\partial x}}\Theta(\bar{\mathcal{L}}\mathrm{d}x) \\ & = -y''^2(1 + y'^2)^{-5/2} + 2y''^2(1 + y'^2)^{-5/2} + 5y'^2y''^2(1 + y'^2)^{-7/2} \\ & \quad -2y'y'''(1 + y'^2)^{-5/2} \\ & = y''^2(1 + y'^2)^{-5/2} + y'(1 + y'^2)^{-7/2}(5y'y''^2 - 2y'''(1 + y'^2)). \end{aligned}$$

Making the change of coordinates to the canonical variables, we obtain

$$H = \tfrac{1}{2}y''p' + y'p = \tfrac{1}{4}p'^2(1 + y'^2)^{5/2} + y'p.$$

In this way, the Hamilton equations (see [7]),

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\partial H}{\partial p} \qquad \frac{\mathrm{d}y'}{\mathrm{d}x} = \frac{\partial H}{\partial p'} \qquad \frac{\mathrm{d}p}{\mathrm{d}x} = -\frac{\partial H}{\partial y} \qquad \frac{\mathrm{d}p'}{\mathrm{d}x} = -\frac{\partial H}{\partial y'}$$

are

$$\frac{\mathrm{d}y}{\mathrm{d}x} = y' \tag{9.1}$$

$$\frac{\mathrm{d}y'}{\mathrm{d}x} = \tfrac{1}{2} p'(1 + y'^2)^{5/2} \tag{9.2}$$

$$\frac{\mathrm{d}p}{\mathrm{d}x} = 0 \tag{9.3}$$

$$\frac{\mathrm{d}p'}{\mathrm{d}x} = -p - \tfrac{5}{4} y' p'^2 (1 + y'^2)^{3/2}. \tag{9.4}$$

Equation (9.1) says nothing but that the solution is the jet of a curve, equation (9.3) states that p is constant along extremals, and the two remaining equations form a system of non-linear ordinary differential equations, whose solutions are the extremals of the variational problem associated with the squared-curvature Lagrangian (cf [6, section 3a]).

As $\bar{\mathcal{L}}$ is trivially invariant with respect to the vector field $\partial/\partial x$, the Hamiltonian must be constant along extremals, as follows from Noether's theorem. If we write p' in terms of y' using equation (9.2) and then substitute it in the expression of the Hamiltonian, we obtain

$$H = \frac{(\mathrm{d}y'/\mathrm{d}x)^2}{(1 + y'^2)^{5/2}} + y'p \qquad \text{or} \qquad \left(\frac{\mathrm{d}y'}{\mathrm{d}x}\right)^2 = (H - py')(1 + y'^2)^{5/2}$$

where H and p are constants. In this way, we have reduced the equations for the extremals of $\bar{\mathcal{L}}$ to a first-order nonlinear ordinary differential equation.

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